

EVOLUTIONS OF S^3 AND $\mathbb{R}P^3$ THAT DESCRIBE EGUCHI-HANSON METRIC AND METRICS OF CONSTANT CURVATURE

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ABSTRACT. In this work we illustrate some well-known facts about the evolution of S^3 under the Ricci flow. The Dirac flow we introduce allows us to describe the 4- dimensional metrics with constant curvature. Another new flow leads to the Eguchi-Hanson metric and can be defined either on metric or on corresponding contact forms.

Keywords: Ricci flow, constant curvature spaces, Eguchi-Hanson metric, Hitchin flow.

1. INTRODUCTION

In this article we interpret some classical 4-dimensional metrics as deformations of the cone over S^3 (or over S^3/\mathbb{Z}_2) generated by a certain evolution equations of the base. We define the Dirac flow on a metric on a sphere S^3

$$\frac{\partial}{\partial t} g_{ij} = \sqrt{\text{Ric}_{ij} - 4K g_{ij}}, \quad i, j \in \{1, 2, 3\} \quad (DF)$$

for $K \in \{-1, 0, +1\}$ and study it in the simplest situation of conformally round metric. If one consider the solution $g = g(t, x)$ of (DF) as a metric on 4-dimensional space then it will be metric of a space with constant curvature. We also define a flow on $\mathbb{R}P^3$, which enables us to describe the Eguchi–Hanson metric

$$\frac{\partial}{\partial t} g_{ij} = \frac{1}{2} \sqrt{\det(\text{Ric})} \text{Ric}_{ij}^{-1}, \quad i, j \in \{1, 2, 3\}.$$

We show that the evolution of $\mathbb{R}P^3$ described by this flow on metric can also be described by the evolution of contact 1-forms given by the Hitchin flow

$$(*\psi)' = d\psi.$$

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We pose the question of constructing new reasonable geometric flows whose particular solutions would yield already known metrics with special properties. Under *reasonable* we understand a parabolic-type equation on the metric with a not-too-complicated dependence of the right-hand side on the Ricci tensor, which plays the role of the Laplacian of the metric. We arrive at this question pursuing rather naive ideas: reversing the direction of time, we can identify the conical singularity with the singularity developed by the Ricci flow in finite time, while the existence of asymptotically locally conical metrics is similar to the existence of ancient solutions.

Onda advances similar ideas [1], asserting in particular that one can describe the Taub–NUT metric as resulting from the action of a Ricci flow (or a backward Ricci flow). He considers the Ricci flow on various three-dimensional Lie groups and checks whether some classical metrics correspond to solutions to the Ricci flow, showing no interest in constructing other flows.

2. THE ROUND THREE-DIMENSIONAL SPHERE IN \mathbb{R}^4

To start with, consider the space $\mathbb{H} = \mathbb{R}^4$. The standard flat metric on \mathbb{R}^4 coincides with the cone metric over $S^3 = SU(2)$. It is known that the Ricci flow $\frac{\partial}{\partial t}g(t) = -2Ric(t)$, whenever it is defined on the Einstein manifold $M(t)$, and on the round sphere in particular, changes only the volume of the manifold. Thus, we can regard \mathbb{R}^4 as the “configuration space” of the Ricci flow on the round three-dimensional spheres changing only their radii. Unfortunately the speed of this change is not appropriate, and the metric induced on \mathbb{R}^4 will not be flat. Roughly speaking, regarding the time coordinate t of some flow which changes only the radius of the sphere as the space coordinate τ , we obtain the space $(\mathbb{R}_+ \times M(\tau))$ (in this example $M(t)$ is a sphere of radius t) with strong restrictions on the curvature, in this case $R_{ijkl} \equiv 0$.

In the case when M is a hypersurface the flow of the mean curvature $\frac{\partial}{\partial t}g(t) = -2b(t)$ is also defined, where $b(t)$ is the second quadratic form depending on the embedding $r : M \rightarrow H$ of one manifold into the other. Henceforth we denote the metric and other tensors on the submanifold M of codimension 1 by \bar{g} , \bar{Ric} , and so on, while tensors on the manifold itself by g , Ric , and so on. The second quadratic form b is defined only on M , and so we write it without the bar.

Viewing S^3 as the Lie group $Sp(1)$ of unit quaternions, we can choose the basis (i, j, k) of three imaginary units in the tangent space $T_1Sp(1)$ at the identity element. Using the multiplication in $Sp(1)$, extend these tangent vectors to three global tangent fields (ξ_1, ξ_2, ξ_3) on the whole

sphere. Associate to them the dual basis in $T^*S^3 = \Lambda^1(S^3)$ consisting of the 1-forms (e^1, e^2, e^3) with $e^i(\xi_j) = \delta_j^i$, usually called the Cartan frame. Consider on \mathbb{R}^4 the flat conical metric

$$g = d\tau^2 + \tau^2((e^1)^2 + (e^2)^2 + (e^3)^2) = d\tau^2 + \bar{g}(\tau) \quad (1)$$

corresponding to the standard spherical coordinate system, where τ plays the role of the radius. Firstly, calculate the second quadratic form of the hypersurface $S^3 \subset \mathbb{R}^4$.

The second quadratic form $(B)|_{ij} = b_{ij}$ of the embedding $r : M \rightarrow \mathbb{R}^n$ of a codimension 1 submanifold is determined from the equalities

$$r_{ij} = b_{ij}\mathbf{m} + \Gamma_{ij}^k r_k, \quad i, j, k = 1, \dots, n-1$$

involving the vectors $r_i = \frac{\partial r}{\partial u^i}$ and $r_{ij} = \frac{\partial^2 r}{\partial u^i \partial u^j}$ of the first and second partial derivatives of r , the unit normal vector \mathbf{m} , and the Christoffel symbols Γ_{ij}^k . Then $b_{ij} = \langle r_{ij}, \mathbf{m} \rangle$.

We can view a sufficiently small neighborhood of every point of $r(M)$ as the zero set of some function $F : \mathbb{R}^n \rightarrow \mathbb{R}$. Then the gradient of this function (with respect to the Euclidean metric) coincides with the normal \mathbf{m} up to a scalar factor; therefore, \mathbb{R}^n includes an open region diffeomorphic to $r(M) \times (0, 1)$. Assume that $\tau = u^0 \in (0, 1)$ and consider a variation r_τ of the embedding r with $\frac{\partial}{\partial \tau} r_\tau = r_0 = \mathbf{m}$. Then $b_{ij} = \Gamma_{ij}^0$. Locally these things are rather well-understood; the main problems arise when we try to extend the coordinate τ : the image $r_\tau(M)$ develops degenerations, which should be controlled. In our simplest example $\tau \in (0, \infty)$, while for $\tau = 0$ the whole sphere collapses to a point, the origin.

In the case of an embedding not into the Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, but rather into an arbitrary Riemannian manifold, we must replace the partial derivatives with covariant derivatives. Then

$$\nabla_i \nabla_j r = b_{ij}\mathbf{m} + \Gamma_{ij}^k \nabla_k r.$$

Observe that the last formula holds only for the coordinate vector fields ∇_i provided that $[\nabla_i, \nabla_j] = 0$, which fails in our example, as $[\xi_i, \xi_{i+1}] = 2\xi_{i+2}$, despite the fact that the sphere embeds into the Euclidean space \mathbb{R}^4 .

It is well-known [2](IV-2) that the Levi-Civita connection is defined as

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + \\ + g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X).$$

Putting in our case $Z = \mathbf{m} = \frac{\partial}{\partial \tau} = \xi_0$, we obtain $2B(X, Y)$ in the left-hand side. Firstly, calculate b_{11} for the flat metric (1). We have

$$2b_{11} = 2g(\nabla_{\xi_1}\xi_1, \xi_0) = -\xi_0 g(\xi_1, \xi_1) = -2\tau$$

because only one term in the right-hand side is not vanishing: the Lie brackets of fields on S^3 are also fields on S^3 and they remain orthogonal to the radial direction ξ_0 , while the fields ξ_i are independent of the radial coordinate τ ; thus, ξ_i commute with ξ_0 . Similarly we calculate two remaining diagonal terms of B . The off-diagonal terms vanish identically because $g(\xi_i, \xi_j) = \delta_{ij}\tau^2$. It is obvious that the restriction of the metric (1) to the three-dimensional sphere of radius τ satisfies the mean curvature flow:

$$(\bar{g}_{ij})'_\tau = \begin{pmatrix} 2\tau & 0 & 0 \\ 0 & 2\tau & 0 \\ 0 & 0 & 2\tau \end{pmatrix} = -2b_{ij}, \quad i, j = 1, 2, 3.$$

Now we would like to check the same equality for the second quadratic form calculated via the Cartan forms. Recall that the connection form of a Riemannian manifold M is the skew-symmetric matrix ω_j^i of 1-forms satisfying

$$d\varepsilon^i = -\omega_j^i \wedge \varepsilon^j,$$

where $\{\varepsilon^1, \dots, \varepsilon^n\}$ is the orthonormal Cartan (co)frame, that is, the basis for $T^*(M)$ consisting of the 1-forms dual to the chosen basis of orthonormal tangent vector fields. In our case $\{\xi_0, \tau^{-1}\xi_1, \tau^{-1}\xi_2, \tau^{-1}\xi_3\}$ constitute an orthonormal frame with respect to the metric g . The differentials of the corresponding 1-forms are

$$\begin{aligned} d\varepsilon^0 &= d(d\tau) = 0, \\ d\varepsilon^1 &= d(\tau e^1) = \frac{1}{\tau}\varepsilon^0 \wedge \varepsilon^1 + \frac{2}{\tau}\varepsilon^2 \wedge \varepsilon^3, \\ d\varepsilon^2 &= d(\tau e^2) = \frac{1}{\tau}\varepsilon^0 \wedge \varepsilon^2 + \frac{2}{\tau}\varepsilon^3 \wedge \varepsilon^1, \\ d\varepsilon^3 &= d(\tau e^3) = \frac{1}{\tau}\varepsilon^0 \wedge \varepsilon^3 + \frac{2}{\tau}\varepsilon^1 \wedge \varepsilon^2. \end{aligned}$$

Since the matrix ω_j^i is skew-symmetric, we easily calculate that

$$-(\omega_j^i)|_{i,j=0..3} = \frac{1}{\tau} \begin{pmatrix} 0 & \varepsilon^1 & \varepsilon^2 & \varepsilon^3 \\ -\varepsilon^1 & 0 & -\varepsilon^3 & \varepsilon^2 \\ -\varepsilon^2 & \varepsilon^3 & 0 & -\varepsilon^1 \\ -\varepsilon^3 & -\varepsilon^2 & \varepsilon^1 & 0 \end{pmatrix}.$$

It is worth noting that we could also calculate the connection form in the old basis $\{e^0, e^1, e^2, e^3\}$, but then we would have to require that the matrix ω_j^i , instead of being skew-symmetric, that is, lying in the algebra $\mathfrak{so}(n)$, belong to the algebra of matrices

$$\{A | AG + GA^T = 0\},$$

where G is the matrix of the metric (1) not equal to a scalar matrix. In other words, while working with Cartan's structure equations, we must always use orthonormal frames and coframes.

Since the connection form is but a generalization of the Christoffel symbols $\omega_j^i = \Gamma_{jk}^i \varepsilon^k$, everything necessary to calculate the second quadratic form is already at hand:

$$b_{jk} = \Gamma_{jk}^0 = \omega_j^0(\tau^{-1}\xi_k) = -\frac{1}{\tau}\delta_{jk}, \quad j, k = 1, 2, 3.$$

Let us verify again, now in the basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, that the standard expanding sphere satisfies the flow of mean curvature:

$$(\bar{g})'_\tau = (\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)'_\tau = (\tau^2(e_1^2 + e_2^2 + e_3^2))'_\tau = \frac{2}{\tau}(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) = -2b.$$

Recall [2](2-5) that the curvature form Ω is the matrix of 2-forms

$$\Omega_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k$$

which generalize the curvature tensor; namely, $\Omega_j^i = \frac{1}{2}R_{jkl}^i \varepsilon^k \wedge \varepsilon^l$. It is not difficult to verify that Ω for the metric (1) vanishes identically; the space \mathbb{R}^4 under consideration is flat. Let us calculate the Ricci tensor \bar{Ric} of \bar{g} . By the symmetries of the curvature tensor,

$$\bar{Ric}_{11} = \bar{R}_{111}^1 + \bar{R}_{121}^2 + \bar{R}_{131}^3 = \bar{R}_{212}^1 + \bar{R}_{313}^1.$$

In order to calculate the last two terms, we have to calculate the curvature form restricted to the sphere

$$\begin{aligned} \bar{\Omega}_2^1 &= d\omega_2^1 + \omega_k^1 \wedge \omega_2^k = d(\frac{1}{\tau}\varepsilon^3) + (-\frac{1}{\tau}\varepsilon^2) \wedge (-\frac{1}{\tau}\varepsilon^1) = -\frac{1}{\tau^2}\varepsilon^0 \wedge \varepsilon^3 \\ &+ \frac{1}{\tau}(\frac{1}{\tau}\varepsilon^0 \wedge \varepsilon^3 + \frac{2}{\tau}\varepsilon^1 \wedge \varepsilon^2) + \frac{1}{\tau^2}\varepsilon^2 \wedge \varepsilon^1 = \frac{1}{\tau^2}\varepsilon^1 \wedge \varepsilon^2 = \frac{1}{2}\bar{R}_{212}^1 \varepsilon^1 \wedge \varepsilon^2 \end{aligned}$$

with summation over $k \in \{1, 2, 3\}$. Then the components of the Ricci tensor are $\bar{Ric}_{11} = \frac{4}{\tau^2} = \bar{Ric}_{22} = \bar{Ric}_{33}$. It may seem here that we arrive at a contradiction since it is known (see for instance [3], 1.159) that, when the metric is multiplied by a constant λ , the Riemann (4,0)-tensor does too, the scalar curvature is multiplied by λ^{-1} , while the Ricci tensor stays unchanged. All our spheres result from a sphere of unit radius by simple homothety, and $\lambda \equiv \tau$. Nevertheless, we find that the Ricci tensor depends on τ . The reason is that the basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ used to calculate the components \bar{Ric}_{ij} also depends on τ . Now we note that in the chosen coordinates the left-hand side of the Ricci flow equation $\bar{g}'_\tau = -2\bar{Ric}$ is a positive definite form, while the right-hand side is negative definite. Therefore, as the space coordinate τ increases, the metric must “decrease” and the corresponding sphere must collapse. Consequently, the standard collapse of the sphere described by the

spherical coordinate system fails to constitute a solution to the Ricci flow.

3. A METRIC WITH ONE FUNCTIONAL PARAMETER

Consider now the metric

$$g = dt^2 + f(t)^2 \left(\sum_{i=1}^3 (e^i)^2 \right) = dt^2 + \bar{g}(t) = \left(\sum_{i=0}^3 (\varepsilon^i)^2 \right). \quad (2)$$

Here we have $\bar{g} = f(t)^2 g_0$, where f is a conformal factor and g_0 is a standard metric on S^3 that does not depend on t anyhow. Although now the 1-forms ε^i , $i = 1, 2, 3$ depend on a function f , we keep the previous notation, being forced to work with orthonormal frames. Then in the basis $\{\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$ the connection form of the metric (2) becomes

$$-\omega = \frac{1}{f} \begin{pmatrix} 0 & f'\varepsilon^1 & f'\varepsilon^2 & f'\varepsilon^3 \\ -f'\varepsilon^1 & 0 & -\varepsilon^3 & \varepsilon^2 \\ -f'\varepsilon^2 & \varepsilon^3 & 0 & -\varepsilon^1 \\ -f'\varepsilon^3 & -\varepsilon^2 & \varepsilon^1 & 0 \end{pmatrix}.$$

An elementary calculation of the curvature form $\bar{\Omega}$ as above yields $\bar{Ric}_{ij} = \frac{4}{f^2} \delta_{ij}$. To satisfy the Ricci flow equation, we should require that

$$\left(\bar{g}_{ij} \right)'_t = \frac{2f'}{f} \delta_{ij} = -\frac{8}{f^2} \delta_{ij} = -2\bar{Ric}_{ij}, \quad i, j = 1, 2, 3; \quad (3)$$

hence, $f(t) = \sqrt{8(t_0 - t)}$. This shows that the radius f of the sphere must depend on time t as the square root, and under the action of the Ricci flow the sphere collapses in finite time, which is a well-known fact. Observe also that for the chosen basis the equation of the mean curvature flow is automatically satisfied; this is the first equality in (3). Therefore, the mean curvature flow in fact becomes a tautology when we work in an invariant basis.

Let us calculate the curvature form of the metric (2) for the ambient space. Now t plays the role of the space coordinate:

$$\begin{aligned} \Omega_1^0 &= -\frac{f''}{f} \varepsilon^0 \wedge \varepsilon^1 = \frac{1}{2} R_{101}^0 \varepsilon^0 \wedge \varepsilon^1, \\ \Omega_2^1 &= \frac{1 - f'^2}{f^2} \varepsilon^1 \wedge \varepsilon^2 = \frac{1}{2} R_{212}^1 \varepsilon^1 \wedge \varepsilon^2. \end{aligned}$$

Accordingly, the Ricci tensor is

$$Ric_{00} = -6\frac{f''}{f}, \quad Ric_{11} = \frac{1}{f^2}(4 - 4f'^2 - 2f''f) = Ric_{22} = Ric_{33},$$

and the scalar curvature is

$$R = \frac{3}{f^2}(4 - 4f'^2 - 4f''f).$$

For $f(t) = f_0(t) = t$ we obtain the flat space \mathbb{R}^4 , for $f(t) = f_{+1}(t) = \sin(t)$ the round sphere S^4 of radius 1, and for $f(t) = f_{-1}(t) = \sinh(t)$ the hyperbolic space H^4 .

However, when we consider the Ricci flow with respect to time τ , for an appropriately chosen dependence between t and τ the sphere S^3 under the action of this flow sweeps a space of constant curvature. For the flat space \mathbb{R}^4 we must put $\frac{dt}{d\tau} = -\frac{4}{t}$ or $\tau = h_0 = \text{const} - \frac{t^2}{8}$. For the sphere S^4 we must put $\tau = h_{+1} = \text{const} + \frac{1}{8}\sin^2 t$, and for the hyperbolic space $\tau = h_{-1} = \text{const} - \frac{1}{8}\sinh^2 t$. This yields rather obvious

Proposition 1. *Consider the round sphere $(S^3, \bar{g}(\tau))$ satisfying the Ricci flow*

$$\frac{\partial}{\partial \tau} \bar{g}(\tau) = -2\bar{Ric}. \quad (RF)$$

If the space coordinate t of the metric (2) is related to the flow time τ by the function h_K then (2) is the metric of a space of constant curvature K for $K \in \{-1, 0, +1\}$.

This claim is obvious. Indeed, consider the Ricci flow as a procedure changing with time the radius of the sphere S^3 . Varying the rate of change of the radius, we can ensure that the space swept by the sphere is of constant curvature.

Now we want to introduce some notations. Denote quotient $\frac{\partial_t \bar{g}}{\bar{g}} = 2\frac{f'}{f} = 2(\ln f)'$ as \dot{f} . And the quotient $\frac{\bar{Ric}}{\bar{g}} = \frac{4}{f^2}$ as f_{Ric} . Note that (DF) will reduce to a single scalar equation because $g_{ij} = \delta_{ij}$ for orthonormal frame $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$. Then the following theorem will be hold.

Theorem 1. *If the round sphere $(S^3, \bar{g}(t))$ changes under the action of the flow defined on the conformal factors*

$$\dot{f} = \sqrt{f_{Ric} - 4K},$$

then the metric g is the metric of the space of constant curvature K for $K \in \{-1, 0, +1\}$ if $f(0) = 0$ and $f'(0) = 1$.

It is easy to see that system of equations from (DF) is simple scalar equation

$$2\frac{f'}{f} = \sqrt{\frac{4}{f^2} - 4K}.$$

It is clear that for suitable initial data, namely $f(0) = 0$ and $f'(0) = 1$, this equation has the solutions f_{-1} , f_0 and f_{+1} . The associated metrics g are of the corresponding curvature K .

One can check that on the class of conformally round metrics this flow and the standard Ricci flow (RF) with respect to time τ related to the space coordinate t by h_K share the solutions describing spaces of constant curvature.

Remind that Dirac equation [4]

$$i\hbar\partial_t = \sqrt{c^2\hbar^2\Delta + m^2c^4}$$

is defined on the differential operators that act on some state functions ψ . It is more or less clear how to define the square root of the operator, and it is not clear how to define the square root of bilinear form G such that \sqrt{G} would be a bilinear form too. Also it is unclear what to do if the expression under the square root will be negative. Remind also that Ricci tensor can be regarded as Laplace operator of the metric in some very symmetric cases, and we think that (DF) can be denoted as 'Dirac flow' although the resemblance is quite superficial.

4. A METRIC WITH TWO FUNCTIONAL PARAMETERS

Consider an embedding $r_t : S^3 \rightarrow \mathbb{R}^4$ inducing the metric

$$g = dt^2 + A_1^2(t)(e^1)^2 + A_2^2(t)((e^2)^2 + (e^3)^2) = dt^2 + \bar{g}(t). \quad (4)$$

Now we define the orthonormal basis in slightly different way:

$$\varepsilon^0 = dt, \quad \varepsilon^1 = A_1(t)e^1, \quad \varepsilon^2 = A_2(t)e^2, \quad \varepsilon^3 = A_2(t)e^3.$$

Then the connection form is

$$-\omega = \begin{pmatrix} 0 & \frac{A'_1}{A_1}\varepsilon^1 & \frac{A'_2}{A_2}\varepsilon^2 & \frac{A'_2}{A_2}\varepsilon^3 \\ -\frac{A'_1}{A_1}\varepsilon^1 & 0 & -\frac{A_1}{A_2^2}\varepsilon^3 & \frac{A_1}{A_2^2}\varepsilon^2 \\ -\frac{A'_2}{A_2}\varepsilon^2 & \frac{A_1}{A_2^2}\varepsilon^3 & 0 & \frac{A_1^2 - 2A_2^2}{A_1A_2^2}\varepsilon^1 \\ -\frac{A'_2}{A_2}\varepsilon^3 & -\frac{A_1}{A_2^2}\varepsilon^2 & -\frac{A_1^2 - 2A_2^2}{A_1A_2^2}\varepsilon^1 & 0 \end{pmatrix}$$

and the curvature form is

$$\begin{aligned} \Omega_1^0 &= \varepsilon^0 \wedge \varepsilon^1 \left[-\frac{A''_1}{A_1} \right] + \varepsilon^2 \wedge \varepsilon^3 \left[-\frac{2A'_1}{A_2^2} + \frac{2A_1A'_2}{A_2^3} \right], \\ \Omega_2^0 &= \varepsilon^0 \wedge \varepsilon^2 \left[-\frac{A''_2}{A_2} \right] + \varepsilon^3 \wedge \varepsilon^1 \left[\frac{A'_1}{A_2^2} - \frac{A_1A'_2}{A_2^3} \right], \\ \Omega_2^1 &= \varepsilon^0 \wedge \varepsilon^3 \left[\frac{A'_1}{A_2^2} - \frac{A_1A'_2}{A_2^3} \right] + \varepsilon^1 \wedge \varepsilon^2 \left[\frac{A_1^2}{A_2^4} - \frac{A'_1A'_2}{A_1A_2} \right], \end{aligned}$$

$$\Omega_3^2 = \varepsilon^0 \wedge \varepsilon^1 \left[\frac{2A_1 A_2'}{A_2^3} - \frac{2A_1'}{A_2^2} \right] + \varepsilon^2 \wedge \varepsilon^3 \left[\frac{4}{A_2^2} - \frac{3A_1^2}{A_2^4} - \frac{A_2'^2}{A_2^2} \right],$$

while the remaining components can be calculated similarly. Observe that for the metric (4) the curvature tensor is non-diagonal. By a diagonal tensor we consider a curvature tensor R_{jkl}^i for which only the sectional curvatures R_{jij}^i can be nonzero. Then the Ricci tensor of the metric \bar{g} has two nontrivial components: $\bar{Ric}_{11} = \bar{R}_{212}^1 + \bar{R}_{313}^1 = 4\frac{A_1^2}{A_2^4}$ and $\bar{Ric}_{22} = \bar{Ric}_{33} = \bar{R}_{212}^1 + \bar{R}_{232}^3 = \frac{4}{A_2^2}(2 - \frac{A_1^2}{A_2^2})$.

The Ricci flow reduces to the system of equations

$$\begin{cases} \frac{A_1'}{A_1} \cdot \frac{dt}{d\tau} = -4\frac{A_1^2}{A_2^4}, \\ \frac{A_2'}{A_2} \cdot \frac{dt}{d\tau} = -\frac{4}{A_2^2}(2 - \frac{A_1^2}{A_2^2}). \end{cases} \quad (5)$$

Assume firstly, as in (3), that $t \equiv \tau$. Put $\alpha = A_1^2$ and $\beta = A_2^2$. The system then becomes

$$\begin{cases} \alpha' = -8\frac{\alpha^2}{\beta^2}, \\ \beta' + 16 = \frac{8\alpha}{\beta}. \end{cases} \quad (6)$$

This system has two obvious solutions. The first corresponds to the case

$$\alpha = \beta = 8(t_0 - t) \quad (\text{N})$$

already discussed. The second solution

$$\alpha = 0, \beta = 16(t_0 - t) \quad (\text{B})$$

corresponds to the collapse of the two-dimensional sphere $S^2 = S^3/S^1$ when the fiber S^1 of the Hopf bundle is collapsed identically. Essentially, the second solution determines the metric (4) on a three-dimensional space.

Remark. We should elucidate our notation. The solution (N) corresponds to the collapsing round three-dimensional sphere. Gibbons and Hawking [5] call solutions with this singularity “nuts” since in this case the whole three-dimensional sphere collapses. A different resolution of the cone singularity, called a “bolt”, resembles the cylinder $S^1 \times \mathbb{R}$ with one coordinate lying on a circle S^1 of small radius and the second coordinate is “long”, that is, lying on the line \mathbb{R} . The second resolution of the singularity amounts to a circle S^1 collapsing in the Hopf bundle with the two-dimensional sphere of radius bounded below as a base. In other words, a two-dimensional sphere plays the role of “long” coordinate, while a one-dimensional circle S^1 (“bolt cross-section”) collapses. Our situation is more degenerate: a one-dimensional circle is collapsed identically, $\alpha \equiv 0$, but in time the two-dimensional sphere collapses

as well, $\beta = 16(t_0 - t)$. Also we can note that in [6] the similar situation appears although the space with singularity has dimension 8. The resolution of "nut"-type is correspond to the space \mathcal{M}_1 and the "bolt"-type singularity — to the space \mathcal{M}_2 .

We can completely integrate (6). From the second equation, we find $\alpha = \frac{1}{8}\beta(\beta' + 16)$. Inserting this into the first equation, we obtain

$$\beta\beta'' + 2\beta'^2 + 48\beta' + 256 = 0.$$

The general solution to this differential equation is

$$-\frac{1}{16}\beta - \frac{\sqrt{2}}{128}c_1 \arctan\left(\frac{4\sqrt{2}\beta}{\sqrt{c_1^2 - 32\beta^2}}\right) = t + c_2, \quad (7)$$

where c_1 and c_2 are constants of integration. When the right-hand side of (7) tends to zero from below, the left-hand side tends to zero from above. Furthermore,

$$-\frac{c_1 + \sqrt{c_1^2 - 32\beta^2}}{\sqrt{c_1^2 - 32\beta^2}} \cdot \frac{\beta'}{16} = 1;$$

hence, for $t \downarrow -c_2 = t_0$ the derivatives satisfy $\beta' \rightarrow -8$ and $\alpha' = \frac{\beta'}{8}(\beta' + 16) + \frac{\beta}{8}\beta'' = \frac{\beta'}{8}(\beta' + 16) + \frac{\beta}{8}\left(\frac{-1}{\beta}\right)(256 + 48\beta' + 2\beta'^2) \rightarrow -8$. In other words, we again recover a well-known fact: under the action of the Ricci flow a distorted three-dimensional sphere collapses to a point in finite time (as a round sphere of infinitesimal radius). We can assert that the solution (B) is nonperturbative, that is,

$$\lim_{t \rightarrow t_0} \alpha'(t) = \lim_{t \rightarrow t_0} \beta'(t) = -8$$

independently of how small $\alpha(0) \neq 0$ is. This agrees with general theory: the Ricci flow makes curvature "uniformly distributed" over all points of the manifold and all tangent directions provided that the initial data is not too bad. Therefore, the sphere completely collapses under the action of the Ricci flow; the Ricci flow cannot decrease just the one-dimensional fiber of the Hopf bundle, keeping the radius of the two-dimensional base bounded below. Thus, we conclude that, using the flows which differ "insignificantly" from the Ricci flow, we have little chance to obtain a description of metrics with resolutions of conical singularity of "bolt"-type.

Consider now the qualitative behavior of solutions. If $\beta(0) > \alpha(0)$ then at the initial moment of time the right-hand sides of (5) are sufficiently small, and α behaves as a constant, while β decreases as $-16t$. When the radius $A_2(t)$ of the sphere is sufficiently close to the radius $A_1(t)$ of the circle, they merge into one solution (N). However,

if $\alpha(0) > \beta(0)$ then, since the right-hand side of the first equation is a negative number of large absolute value, it follows that α quite quickly becomes equal to β , and they merge again.

Remark. In computer simulations, due to round-off errors sometimes (that is, not in all simulations) the solution to (5) extends beyond the singularity time t_0 . The function α extends by zero, while the function β becomes equal to $16(t_0 - t)$ and so goes into the negative region, and the metric (4) ceases to be Riemannian; moreover, it ceases to be a metric on a four-dimensional manifold. Nevertheless, we can assert that the solution (N) becomes the solution (B) by passing through the singularity. It is not clear whether we can find a meaningful interpretation of this effect or this is just an artifact of simulations.

This transformation of solution (N) to (B) has nothing to do with the resolution of singularity using the normalization of the Ricci flow. Recall that the normalized Ricci flow is

$$g' = -2Ric + \frac{2}{n}\mathfrak{R}g,$$

where n is the dimension of the manifold and \mathfrak{R} is the average scalar curvature. In our case the normalized flow is

$$\begin{cases} A_1' = -\frac{16}{3} \frac{A_1}{A_2^2} (A_1^2 - A_2^2), \\ A_2' = \frac{8}{3} \cdot \frac{A_1^2 - A_2^2}{A_2^3} \end{cases}$$

since the functions A_1 and A_2 depend only on time and it is unnecessary to average the curvature over the sphere. We can easily solve these equations; more so when we remember that the action of the normalized Ricci flow preserves the volume $\sqrt{\det(\bar{g})} = A_1 A_2^2$. From the right-hand side of this system it is not difficult to see also that, as $t \rightarrow \infty$, both functions tend to the same constant. Hence, in infinite time we again obtain a round sphere of constant radius.

We can define the decoupled Ricci flow (5) by replacing the left-hand side while taking different derivatives with respect to time in different directions:

$$\begin{cases} \frac{A_1'}{A_1} \cdot r^{-1} = -4 \frac{A_1^2}{A_2^4}, \\ \frac{A_2'}{A_2} \cdot s^{-1} = -\frac{4}{A_2^2} (2 - \frac{A_1^2}{A_2^2}), \end{cases} \quad (5')$$

where r and s are two functions of t . It obviously makes little sense to consider arbitrary r and s ; therefore, we try to find some geometrically meaningful particular cases. Moreover, we can consider an equation of even more general form:

$$\mathfrak{D}g = R(g),$$

where $\mathfrak{D}g = (\frac{1}{r}\frac{\partial}{\partial t}g|_{\mathcal{V}}, \frac{1}{s}\frac{\partial}{\partial t}g|_{\mathcal{H}})$ is the derivative of the metric determined by the natural Riemannian decomposition into vertical and horizontal fibers of the space on which the considered metric is defined. In many cases the horizontal subbundle may also admit a decomposition, and the number of components of the derivative \mathfrak{D} may increase.

Recall that a metric is called anti-self-dual whenever its connection form satisfies

$$\omega_j^i = -\frac{1}{2}\varepsilon_{ijkl}\omega_l^k,$$

where ε is the Levi-Civita symbol. The Ricci tensor of an anti-self-dual metric, as well as of a self-dual one, automatically vanishes. In the 4-dimensional case the anti-self-duality reduces to the pair of equations

$$\omega_1^0 = -\omega_3^2, \quad \omega_2^0 = \omega_3^1$$

or, in our case,

$$\frac{A_1'}{A_1} = -\frac{A_1^2 - 2A_2^2}{A_1A_2^2}, \quad \frac{A_2'}{A_2} = \frac{A_1}{A_2^2} \quad (8).$$

It is well-known that these equations are integrable; this is how the classical Eguchi–Hanson metric [7]

$$ds^2 = [1 - (a/r)^4]^{-1}dr^2 + r^2((e^2)^2 + (e^3)^2) + r^2[1 - (a/r)^4](e^1)^2$$

was found. This metric was the first metric with holonomy group $SU(2)$ and an explicit ansatz in elementary functions. Recall that as r tends to a , the Eguchi–Hanson metric has a singularity of type “bolt”, while for sufficiently large R the set $\{r = R\}$ is homeomorphic to $\mathbb{R}P^3$.

It is not difficult to observe that, even though the right-hand sides of (8) involve the components of the connection form, they can be expressed in terms of the components of the Ricci tensor:

$$-\frac{A_1^2 - 2A_2^2}{A_1A_2^2} = \frac{1}{2}\bar{Ric}_{22}(\bar{Ric}_{11})^{-1/2},$$

$$\frac{A_1}{A_2^2} = \frac{1}{2}(\bar{Ric}_{11})^{1/2}.$$

Thus, (8) is equivalent to the flow

$$\frac{\partial}{\partial t}\bar{g}_{ij} = \frac{1}{2}\sqrt{\det(\bar{Ric})}\bar{Ric}_{ij}^{-1}, \quad (9)$$

where \bar{Ric}_{ij}^{-1} are the components of the 3×3 -matrix of the Ricci tensor in orthonormal frame ε^j . We obtain a non-splitting flow on the three-dimensional sphere, that is, a flow which we can define on the total space of the Hopf bundle instead of defining it separately on the fiber S^1 and base S^2 . Unfortunately, the resulting flow has an extremely unpleasant right-hand side. This confirms our suggestion that a metric

with singularity of type “bolt” cannot be described by a geometric flow with good right-hand side. Observe also that for $A_1 = A_2$ the function $f(t)$ considered above is a solution, while the corresponding metric g is the metric of a flat space. This follows immediately from the fact that the metric of a flat space is anti-self-dual.

Theorem 2. *Consider the projective space $\mathbb{R}P^3 = S^3/\mathbb{Z}_2$ with the metric*

$$A_1^2(t)(e^1)^2 + A_2^2(t)((e^2)^2 + (e^3)^2)$$

changing under the action of the flow (9). If $A_1(0) = 0$, $A_1'(0) = 2$, and $A_2(0) = a$, then the corresponding metric (4) is isometric to the Eguchi–Hanson metric.

As flow (9) has such terrible right-hand side it would be natural to consider the evolution of some structures that respect the Hopf fibration. We remind that the 1-form ψ on a $(2n+1)$ -dimensional manifold called a contact form if $\psi \wedge (d\psi)^n \neq 0$. It is easy to check that

$$\varepsilon^1 \wedge \bar{d}\varepsilon^1 = 2\frac{A_1}{A_2^2}\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3,$$

where \bar{d} is a differential acting on S^3 , that does not depend on t and has no connection with Dolbeault theory. Also

$$\varepsilon^2 \wedge \bar{d}\varepsilon^2 = 2\frac{1}{A_1}\varepsilon^1 \wedge \varepsilon^2 \wedge \varepsilon^3.$$

If one defines the following flow on the contact structures

$$(*\psi)' = \bar{d}\psi, \tag{10}$$

where $*$ is the Hodge operator (with respect to orthonormal frame $\{\varepsilon^1, \varepsilon^2, \varepsilon^3, \}$), then for $\psi = \varepsilon^1$ one will get exactly the second equation from (8). And for $\psi = \varepsilon^2$ one will get

$$\left(\frac{A_1'}{A_1} + \frac{A_2'}{A_2}\right)\varepsilon^3 \wedge \varepsilon^1 = \frac{2}{A_1}\varepsilon^3 \wedge \varepsilon^1.$$

That equation combined with one for $\psi\varepsilon^1$ will give the first equation from (8). Obviously for $\psi = \varepsilon^3$ one will have exactly the same. So we have the following

Theorem 3. *Consider the projective space $\mathbb{R}P^3 = S^3/\mathbb{Z}_2$ with the metric*

$$A_1^2(t)(e^1)^2 + A_2^2(t)((e^2)^2 + (e^3)^2)$$

associated with 3 contact 1-forms $\varepsilon^1 = A_1(t)e^1$, $\varepsilon^2 = A_2(t)e^2$, $\varepsilon^3 = A_2(t)e^3$, that are changed under the action of the flow (10). Then for

appropriate initial data the metric

$$dt^2 + A_1^2(t)(e^1)^2 + A_2^2(t)((e^2)^2 + (e^3)^2)$$

is isometric to the Eguchi–Hanson metric.

Remark on connection with G_2 .

We must mention that the flow (10) was introduced by Hitchin (see for example [8]) in order to obtain an evolution of the 3-form ϕ_0 that defines the G_2 -structure on 7-dimensional manifold. He proved that the solution ϕ_t of the system (10) define a G_2 -structure for sufficiently small times.

Also we observe that the equations (8) with some trivial changes were written in [9]. In that paper there was an attempt to construct a metric with G_2 -holonomy on deformed cone over twistor space of 7-dimensional 3-Sasakian manifold.

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